Problem 2.24

Consider a sphere (diameter D, density ϱ_{sph}) falling through air (density ϱ_{air}) and assume that the drag force is purely quadratic. **(a)** Use Equation (2.84) from Problem 2.4 (with $\kappa = 1/4$ for a sphere) to show that the terminal speed is

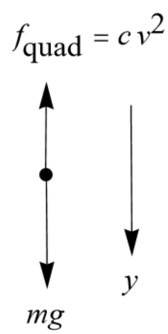
$$v_{\text{ter}} = \sqrt{\frac{8}{3}Dg\frac{\varrho_{\text{sph}}}{\varrho_{\text{air}}}}. \tag{2.88}$$

(b) Use this result to show that of two spheres of the same size, the denser one will eventually fall faster. **(c)** For two spheres of the same material, show that the larger will eventually fall faster.

Solution

Part (a)

Draw a free-body diagram for a mass falling down in a medium with quadratic air resistance. Let the positive y-direction point downward.



Apply Newton's second law in the y-direction.

$$\sum F_y = ma_y$$

Let $v_y = v$ to simplify the notation.

$$mg - cv^2 = m\frac{dv}{dt}$$

The terminal speed occurs when the velocity reaches equilibrium.

$$mg - cv_{\text{ter}}^2 = m(0)$$

Solve for the terminal velocity.

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}$$

By Equation (2.84) on page 73, $c = \kappa \varrho_{\text{air}} A$, where $\kappa = 1/4$ and A is the area of the sphere's cross-section: $A = \pi(D/2)^2 = \pi D^2/4$.

$$\begin{aligned} v_{\text{ter}} &= \sqrt{\frac{mg}{\kappa \varrho_{\text{air}} A}} \\ &= \sqrt{\frac{mg}{\frac{1}{4}\varrho_{\text{air}} \frac{\pi D^2}{4}}} \\ &= \sqrt{\frac{16mg}{\pi \varrho_{\text{air}} D^2}} \\ &= \sqrt{\frac{16(\varrho_{\text{sph}} V)g}{\pi \varrho_{\text{air}} D^2}} \\ &= \sqrt{\frac{16\varrho_{\text{sph}} \left[\frac{4}{3}\pi \left(\frac{D}{2}\right)^3\right] g}{\pi \varrho_{\text{air}} D^2}} \\ &= \sqrt{\frac{8}{3} Dg \frac{\varrho_{\text{sph}}}{\varrho_{\text{air}}}} \end{aligned}$$

Part (b)

Suppose there are two spheres with the same size (both with diameter D) and that sphere 1 is denser than sphere 2. Then the terminal speeds of these two spheres are

$$\begin{cases} v_{\text{ter1}} = \sqrt{\frac{8}{3} Dg \frac{\varrho_{\text{sph1}}}{\varrho_{\text{air}}}} \\ \\ v_{\text{ter2}} = \sqrt{\frac{8}{3} Dg \frac{\varrho_{\text{sph2}}}{\varrho_{\text{air}}}} \end{cases} \quad \Rightarrow \quad \frac{v_{\text{ter1}}}{v_{\text{ter2}}} = \frac{\sqrt{\frac{8}{3} Dg \frac{\varrho_{\text{sph1}}}{\varrho_{\text{air}}}}}{\sqrt{\frac{8}{3} Dg \frac{\varrho_{\text{sph2}}}{\varrho_{\text{air}}}}} = \sqrt{\frac{\varrho_{\text{sph1}}}{\varrho_{\text{sph2}}}}.$$

Since sphere 1 is denser than sphere 2, $\varrho_{\text{sph1}} > \varrho_{\text{sph2}}$, so

$$\frac{v_{\text{ter1}}}{v_{\text{ter2}}} = \sqrt{\frac{\varrho_{\text{sph1}}}{\varrho_{\text{sph2}}}} > 1.$$

This means the terminal speed of sphere 1 is larger than that of sphere 2.

$$v_{\text{ter1}} > v_{\text{ter2}}$$

Part (c)

Suppose there are two spheres with the same material (both with density ϱ_{sph}) and that sphere 1 is larger than sphere 2. Then the terminal speeds of these two spheres are

$$\begin{cases} v_{\text{ter1}} = \sqrt{\dfrac{8}{3} D_1 g \dfrac{\varrho_{\text{sph}}}{\varrho_{\text{air}}}} \\ \\ v_{\text{ter2}} = \sqrt{\dfrac{8}{3} D_2 g \dfrac{\varrho_{\text{sph}}}{\varrho_{\text{air}}}} \end{cases} \quad \Rightarrow \quad \frac{v_{\text{ter1}}}{v_{\text{ter2}}} = \frac{\sqrt{\frac{8}{3} D_1 g \frac{\varrho_{\text{sph}}}{\varrho_{\text{air}}}}}{\sqrt{\frac{8}{3} D_2 g \frac{\varrho_{\text{sph}}}{\varrho_{\text{air}}}}} = \sqrt{\frac{D_1}{D_2}}.$$

Since sphere 1 is larger than sphere 2, $D_1 > D_2$, so

$$\frac{v_{\text{ter1}}}{v_{\text{ter2}}} = \sqrt{\frac{D_1}{D_2}} > 1.$$

This means the terminal speed of sphere 1 is larger than that of sphere 2.

$$v_{\text{ter1}} > v_{\text{ter2}}$$